GEOMETRY — SEPTEMBER 2025

MID-SEMESTER EXAM — 100 POINTS

Notes.

(a) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) We use $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers, $I = [0, 1]$, the unit interval in $\mathbb{R}$.

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1. [5 + 5 + 12 = 22 Points] Let X be a topological space.
   (i) Define what it means for X to be contractible.
   (ii) Define what it means for X to be simply connected.
   (iii) Prove that if X is contractible, then it is simply connected.

2. [22 Points] Prove the theorem on lifting of paths for covering maps:
Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map. Let $f\colon I \to B$ be any map sending 0 to b_0. Then there exists a unique lift of f to a map $\tilde{f}\colon I \to E$ such that $\tilde{f}(0) = e_0$.

3. [4 + 7 + 9 + 4 = 24 Points] Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map.
   (i) Define the lifting correspondence $\pi_1(B, b_0) \to p^{-1}(b_0)$.
   (ii) Prove that the lifting correspondence is surjective iff E is path-connected.
   (iii) Prove that the lifting correspondence is bijective iff E is simply connected.
   (iv) Prove that the natural map $p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective.

4. [20 Points] Group the following spaces according to their homotopy type, i.e., according to whether they are homotopy equivalent or not. You must justify your answer. You may quote any theorem proved in class.
   (i) The annulus $\{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ in the complex plane $\mathbb{C}$.
   (ii) $\mathbb{R}^3$ minus the x-axis.
   (iii) The solid torus $B^2 \times S^1$.
   (iv) The hollow torus $S^1 \times S^1$.
   (v) The punctured solid sphere $B^3 \setminus \{p\}$ where p is a point in the interior.

5. [6 + 6 = 12 Points] Let G_1 be a cyclic group of order two with generator a and G_2 a cyclic group of order 3 with generator b.
   (i) Write down all reduced words in G_1 and G_2 of length 5.
   (ii) Define a surjective homomorphism $G_1 * G_2 \to G_2$ and describe it in terms of where each element goes to.